The colourful simplicial depth conjecture

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THE COLOURFUL SIMPLICIAL DEPTH CONJECTURE

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Abstract. Given $d+1$ sets of points, or colours, $\mathbf{S}_1,\ldots,\mathbf{S}_{d+1}$ in $\mathbb{R}^d$, a *colourful simplex* is a set $T \subseteq \bigcup_{i=1}^{d+1} \mathbf{S}_i$ such that $|T \cap \mathbf{S}_i| \leq 1$, for all $i \in \{1,\ldots,d+1\}$. The colourful Carathéodory theorem states that, if $\mathbf{0}$ is in the convex hull of each $\mathbf{S}_i$, then there exists a colourful simplex T containing $\mathbf{0}$ in its convex hull. Deza, Huang, Stephen, and Terlaky (*Colourful simplicial depth*, Discrete Comput. Geom., **35**, 597–604 (2006)) conjectured that, when $|\mathbf{S}_i| = d+1$ for all $i \in \{1,\ldots,d+1\}$, there are always at least d^2+1 colourful simplices containing $\mathbf{0}$ in their convex hulls. We prove this conjecture via a combinatorial approach.

1. Introduction

A *colourful point configuration* is a collection of $d+1$ sets of points $\mathbf{S}_1,\ldots,\mathbf{S}_{d+1}$ in $\mathbb{R}^d$. A *colourful simplex* is a subset T of $\bigcup_{i=1}^{d+1} \mathbf{S}_i$ such that $|T \cap \mathbf{S}_i| \leq 1$. The colourful Carathéodory theorem, proved by Bárány in 1982 [1], states that, given a colourful point configuration $\mathbf{S}_1,\ldots,\mathbf{S}_{d+1}$ in $\mathbb{R}^d$ such that $\mathbf{0} \in \bigcap_{i=1}^{d+1} \operatorname{conv}(\mathbf{S}_i)$, there exists a colourful simplex T containing $\mathbf{0}$ in its convex hull. In the same paper, Bárány uses this theorem combined with Tverberg's theorem to give a bound on simplicial depth. His argument motivated the following question: how many colourful simplices, at least, contain $\mathbf{0}$ in their convex hulls?

Let $\mu(d)$ denote the minimal number of colourful simplices containing $\mathbf{0}$ in their convex hulls over all colourful point configurations $\mathbf{S}_1,\ldots,\mathbf{S}_{d+1}$ in $\mathbb{R}^d$ such that $\mathbf{0} \in \operatorname{conv}(\mathbf{S}_i)$ and $|\mathbf{S}_i| = d+1$ for $i = 1,\ldots,d+1$. The colourful Carathéodory theorem states that $\mu(d) \geq 1$. The quantity $\mu(d)$ has been investigated by Deza, Huang, Stephen, and Terlaky [3]. They proved that $2d \leq \mu(d) \leq d^2+1$ and conjectured that $\mu(d) = d^2+1$. Later Bárány and Matoušek [2] proved that $\mu(d) \geq \max\left(3d, \left\lceil \frac{d(d+1)}{5} \right\rceil\right)$ for $d \geq 3$, Stephen and Thomas [6] proved that $\mu(d) \geq \left\lfloor \frac{(d+2)^2}{4} \right\rfloor$, and Deza, Stephen, and Xie [4] showed that $\mu(d) \geq \left\lceil \frac{(d+1)^2}{2} \right\rceil$. Deza, Meunier, and Sarrabezolles [5] improved the bound to $\frac{1}{2}d^2+\frac{7}{2}d-8$ for $d \geq 4$. This latter result was obtained using a combinatorial generalization of the colourful point configurations suggested by Bárány and known as *octahedral systems*, see [4].

We use this combinatorial approach to prove the conjecture.

Theorem 1. *The equality $\mu(d) = d^2+1$ holds for every integer $d \geq 1$.*

The outline of the paper goes as follows. Section 2 is divided into two parts. First we define the octahedral systems and show their link with the colourful point configurations. Second, we introduce one of our main tools: the decomposition of an octahedral system over some elementary octahedral systems called umbrellas. Section 3 is devoted to the proof of Theorem 1.

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2. Preliminaries

2.1. **Octahedral systems.** Let $V_1, \ldots, V_n$ be n pairwise disjoint finite sets, each of size at least 2. An *octahedral system* is a set $\Omega \subseteq V_1 \times \cdots \times V_n$ satisfying the *parity condition*: the cardinality of $\Omega \cap (X_1 \times \cdots \times X_n)$ is even if $X_i \subseteq V_i$ and $|X_i| = 2$ for all $i \in \{1, \ldots, n\}$. We use the terminology of hypergraphs to describe an octahedral system: the sets V_i are the *classes*, the elements in V_i are the *vertices*, and the n-tuples in $V_1 \times \cdots \times V_n$ are the *edges*. An edge whose ith component is a vertex $x \in V_i$ is *incident with the vertex* x, and conversely. A vertex x incident with no edges is *isolated*. A class V_i is *covered* if each vertex of V_i is incident with at least one edge. Finally, the set of edges incident with x is denoted by $\delta_\Omega(x)$ and the *degree of* x, denoted by $\deg_\Omega(x)$, refers to $|\delta_\Omega(x)|$.

Lemma 1. *In every nonempty octahedral system, at least one class is covered.*

Proof. Consider an octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$. Suppose that no classes are covered. There is at least one isolated vertex x_i in each V_i. Hence, if there were an edge $(y_1, \ldots, y_n)$ in Ω, then the parity condition would not be satisfied for $X_i = \{x_i, y_i\}$. □

Given a colourful point configuration $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$, the Octahedron Lemma [2, 3] states that, for any $\mathbf{S}'_1 \subseteq \mathbf{S}_1, \ldots, \mathbf{S}'_{d+1} \subseteq \mathbf{S}_{d+1}$, with $|\mathbf{S}'_1| = \cdots = |\mathbf{S}'_{d+1}| = 2$, the number of colourful simplices generated by $\bigcup_{i=1}^{d+1} \mathbf{S}'_i$ and containing $\mathbf{0}$ in their convex hulls is even. The hypergraph over $V_1 \times \cdots \times V_n$ where V_i is identified with $\mathbf{S}_i$ and whose edges are identified with the colourful simplices containing $\mathbf{0}$ in their convex hulls is therefore an octahedral system. Furthermore, a strengthening of the colourful Carathéodory Theorem, given in [1], states that if $\mathbf{0} \in \bigcap_{i=1}^{d+1} \operatorname{conv}(\mathbf{S}_i)$, then each point of the colourful point configuration is in some colourful simplices containing $\mathbf{0}$ in their convex hulls. Hence, in an octahedral system Ω arising from such a colourful point configuration, each class V_i is covered.

2.2. **Decompositions.** The following proposition, proved in [5], states that the set of all octahedral systems is stable under the "symmetric difference" operation.

Proposition 1. *Let Ω and Ω' be two octahedral systems over the same vertex set. $\Omega \triangle \Omega'$ is an octahedral system.*

Proof. Let $\Omega'' = \Omega \triangle \Omega'$. As Ω'' is a subset of $V_1 \times \cdots \times V_n$, we simply check that the parity condition is satisfied. Consider $X_1 \subseteq V_1, \ldots, X_n \subseteq V_n$ with $|X_i| = 2$ for $i = 1, \ldots, n$. We have

$$|\Omega'' \cap (X_1 \times \cdots \times X_n)| = |\Omega \cap (X_1 \times \cdots \times X_n)| + |\Omega' \cap (X_1 \times \cdots \times X_n)| - 2|\Omega \cap \Omega' \cap (X_1 \times \cdots \times X_n)|.$$

All the terms of the sum are even, which allows to conclude. □

We now present a family of specific octahedral systems we call *umbrellas*. An umbrella U is a set of the form $\{x^{(1)}\} \times \cdots \times \{x^{(i-1)}\} \times V_i \times \{x^{(i+1)}\} \times \cdots \times \{x^{(n)}\}$, with $x^{(j)} \in V_j$ for $j \neq i$. The class V_i covered in U is called its *colour*. $T = (x^{(1)}, \ldots, x^{(i-1)}, x^{(i+1)}, \ldots, x^{(n)})$ is its *transversal*. An umbrella is clearly an octahedral system over $V_1 \times \cdots \times V_n$ and we have the following proposition.

Proposition 2. *Two umbrellas of the same colour have an edge in common if and only if they are equal.*

Proof. An umbrella is entirely determined by its colour V_i and its transversal T. Therefore, if two umbrellas of the same colour have an edge in common, they necessarily have the same transversal, which implies that they are equal. $\square$

It was implicitly proved in Section 3 of [5] that any octahedral system can be described as a symmetric difference of umbrellas In this paper, we describe an octahedral system as a symmetric difference of other octahedral systems to bound its cardinality.

Consider a nonempty octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$ with $|V_i| = n$ for all $i \in \{1, \ldots, n\}$. Denote by i_1 the smallest $i \in \{1, \ldots, n\}$ such that V_i is covered in Ω and order the vertices $\{x_1, \ldots, x_n\}$ of V_{i_1} by increasing degree: $\deg_\Omega(x_1) \leq \cdots \leq \deg_\Omega(x_n)$. We define $\mathcal{U}$ to be the set of umbrellas of colour V_{i_1} containing an edge of Ω incident with x_1 and $W = \triangle_{U \in \mathcal{U}} U$. Let Ω_j be the set of all edges in $\Omega \triangle W$ incident with x_j. Formally,

$$\mathcal{U} = \{U : U \text{ umbrella of colour } V_{i_1} \text{ and } U \cap \delta_\Omega(x_1) \neq \emptyset\} \text{ and } \Omega_j = \delta_{\Omega \triangle W}(x_j).$$

Note that $|\mathcal{U}| = \deg_\Omega(x_1)$. In the remaining of the paper we refer to $(\mathcal{U}, \Omega_2, \ldots, \Omega_n)$ as a *suitable decomposition.*

Lemma 2. *Let $(\mathcal{U}, \Omega_2, \ldots, \Omega_n)$ be suitable decomposition and let $W = \triangle_{U \in \mathcal{U}} U$. We have*

(i) $\Omega_j \cap \Omega_\ell = \emptyset$*, for all $j \neq \ell$ (they have no edge in common),*
(ii) $\Omega = W \triangle \Omega_2 \triangle \cdots \triangle \Omega_n$*,*
(iii) Ω_j *is an octahedral system, for all j,*
(iv) $\deg_\Omega(x_j) \geq \max(|\mathcal{U}|, |\Omega_j| - |\Omega_j \cap W|)$ *for all j.*
(v) *If V_i is not covered in Ω, then V_i is not covered in $\Omega \triangle W$ and V_i is covered in no Ω_j.*

The terminology suitable decomposition is due to point (ii) of Lemma 2.

Proof of Lemma 2. We first prove (i). The i_1th component of any edge in Ω_j is x_j. Therefore, Ω_j and Ω_ℓ have no edge in common if $j \neq \ell$.

We then prove (ii). There are exactly $\deg_\Omega(x_1)$ umbrellas of colour V_{i_1} containing an edge of Ω incident with x_1. As W is the symmetric difference of these umbrellas, x_1 is isolated in $\Omega \triangle W$. Thus, $\Omega_2, \ldots, \Omega_n$ form a partition of the edges in $\Omega \triangle W$ and $\Omega \triangle W = \Omega_2 \triangle \cdots \triangle \Omega_n$. Taking the symmetric difference of this equality with W we obtain $\Omega = W \triangle \Omega_2 \triangle \cdots \triangle \Omega_n$.

We now prove (iii). By definition, the Ω_j's are subsets of $V_1 \times \cdots \times V_n$. It remains to prove that they satisfy the parity condition. Consider $X_i \subseteq V_i$ with $|X_i| = 2$ for $i = 1, \ldots, n$. If X_{i_1} does not contain x_j, there are no edges in Ω_j induced by $X_1 \times \cdots \times X_n$. If X_{i_1} contains x_j, the edges in Ω_j induced by $X_1 \times \cdots \times X_n$ are the ones induced by $X_1 \times \cdots \times X_{i_1 - 1} \times \{x_j\} \times X_{i_1+1} \times \cdots \times X_n$. As x_1 is isolated in $\Omega \triangle W$, those edges are exactly the edges in $\Omega \triangle W$ induced by $X_1 \times \cdots \times X_{i_1 - 1} \times \{x_1, x_j\} \times X_{i_1+1} \times \cdots \times X_n$. According to Proposition 1, W is an octahedral system and $\Omega \triangle W$ as well, hence there is an even number of edges.

We prove (iv). We have $|\mathcal{U}| = \deg_\Omega(x_1) \leq \deg_\Omega(x_j)$ for all $j \in \{1, \ldots, n\}$. Furthermore, by definition of the symmetric difference, we have $(\Omega_2 \triangle \cdots \triangle \Omega_n) \setminus W \subseteq \Omega$. This inclusion becomes $(\Omega_2 \setminus W) \triangle \cdots \triangle (\Omega_n \setminus W) \subseteq \Omega$. As two Ω_ℓ's share no edges, $\Omega_j \setminus W \subseteq \Omega$ and thus $\Omega_j \setminus W \subseteq \delta_\Omega(x_j)$ for all $j \in \{2, \ldots, n\}$. We obtain

$$|\Omega_j| - |\Omega_j \cap W| \leq \deg_\Omega(x_j).$$

Finally to prove (v) it suffices to prove that a class V_i not covered in Ω remains not covered in $\Omega \triangle W$. Indeed, if a class is covered in an Ω_j, it is also covered in $\Omega \triangle W$, as no two Ω_ℓ's have an edge in common. Consider V_i not covered in Ω. There is a vertex $x \in V_i$ incident with no edges in Ω. In particular, there are no edges in Ω incident with x_1 and x. Therefore, the umbrellas in $\mathcal{U}$, which are defined by the edges incident with x_1, contain no edges incident with x. Hence, x is isolated in $W = \triangle_{U\in\mathcal{U}} U$ and in Ω. Finally, x remains isolated in $\Omega \triangle W$. □

Unlike the suitable decomposition of Ω, which is a decomposition over general octahedral systems, the decomposition given in the following lemma is over umbrellas.

Lemma 3. *Consider an octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$ with $|V_i| = n$ for all $i \in \{1, \ldots, n\}$. There exists a set of umbrellas $\mathcal{D}$, such that $\Omega = \triangle_{U\in\mathcal{D}} U$ and such that the following implication holds:*

$$V_i \text{ is the colour of some } U \in \mathcal{D} \Longrightarrow V_i \text{ is covered in } \Omega.$$

Proof. The proof works by induction on the number of covered classes in Ω. If no classes are covered, then, according to Lemma 1, Ω is empty.

Suppose now that k classes are covered, with $k \geq 1$, and consider a suitable decomposition $(\mathcal{U}, \Omega_2, \ldots, \Omega_n)$ of Ω. Denote by W the symmetric difference $W = \triangle_{U\in\mathcal{U}} U$. According to Proposition 1, W is an octahedral system, and so is $\Omega \triangle W$. There are stricly fewer covered classes in $\Omega \triangle W$ than in Ω. Indeed, in $\Omega \triangle W$, the class V_{i_1} is no longer covered, since x_1 is isolated, and according to (v) of Lemma 2, a class not covered in Ω remains not covered in $\Omega \triangle W$. By induction, there exists a set $\mathcal{D}'$ of umbrellas such that $\Omega \triangle W = \triangle_{U\in\mathcal{D}'} U$, and such that if there is an umbrella of colour V_i in $\mathcal{D}'$, then V_i is covered in $\Omega \triangle W$. As the umbrellas in $\mathcal{D}'$ are not of colour V_{i_1}, we have $\mathcal{U} \cap \mathcal{D}' = \emptyset$. Therefore, $\Omega = (\triangle_{U\in\mathcal{U}} U) \triangle (\triangle_{U\in\mathcal{D}'} U)$ and the set $\mathcal{D} = \mathcal{U} \cup \mathcal{D}'$ satisfies the statement of the lemma. □

3. Proof of the main result

The following theorem gives a general lower bound on the cardinality of an octahedral system. Our main theorem is a corollary of it.

Theorem 2. *Let $\Omega \subseteq V_1 \times \cdots \times V_n$ be an octahedral system with $|V_1| = \cdots = |V_n| = n \geq 2$. If $k \geq 1$ classes among the V_i's are covered, then*

$$|\Omega| \geq k(n-2) + 2.$$

Before proving this theorem, we show how the main theorem can be deduced from it.

Proof of Theorem 1. The inequality $\mu(d) \leq d^2 + 1$ is proved in [3]. Let $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ be a colourful point configuration in $\mathbb{R}^d$. As explained in Section 2.1, the set $\Omega \subseteq V_1 \times \cdots \times V_{d+1}$, with $V_i = \mathbf{S}_i$ for $i = 1, \ldots, d+1$ and whose edges correspond to the colourful simplices containing $\mathbf{0}$ in their convex hulls, is an octahedral system. According to [1, Theorem 2.3.], all the classes are covered in this octahedral system. Applying Theorem 2 with $k = n = d+1$ gives the lower bound: $\mu(d) \geq d^2 + 1$. □

The remainder of the section is devoted to the proof of Theorem 2. The proof distinguishes two cases, corresponding to the following Propositions 3 and 4. We first prove these propositions.

Proposition 3. *Consider an octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$ with $|V_i| = n$ for all $i \in \{1, \ldots, n\}$ and a class V_i covered in Ω. If Ω can be written as a symmetric difference of umbrellas, none of them being of colour V_i, then $|\Omega| \geq n^2$.*

Proof. Let $\mathcal{D}$ be a set of umbrellas such that there are no umbrellas of colour V_i in $\mathcal{D}$ and $\Omega = \triangle_{U\in\mathcal{D}}U$. Denote by $y_1,\ldots,y_n$ the vertices of V_i, and by $\mathcal{Q}_j$ the set of umbrellas in $\mathcal{D}$ incident with y_j for each $j\in\{1,\ldots,n\}$. As $\mathcal{D}$ does not contain any umbrellas of colour V_i, the umbrellas in $\mathcal{Q}_j$ all have transversals with ith component equal to y_j. Denote by Q_j the symmetric difference of the umbrellas in $\mathcal{Q}_j$. We have that Q_j is an octahedral system, according to Proposition 1, and that $\delta_\Omega(x_j) = Q_j$, $Q_j \neq \emptyset$, and $Q_j \cap Q_\ell = \emptyset$ for all $j \neq \ell$. According to Lemma 1, at least one class is covered in Q_j and hence $|Q_j| \geq n$. Therefore, we have

$$|\Omega| = \sum_{j=1}^{n} \deg_\Omega(x_j) = \sum_{j=1}^{n} |Q_j| \geq n^2$$

□

Proposition 4. *Consider an octahedral system $\Omega \subseteq V_1\times\cdots\times V_n$ with $|V_i| = n$ for all $i\in\{1,\ldots,n\}$ and a suitable decomposition $(\mathcal{U},\Omega_2,\ldots,\Omega_n)$ of Ω. Consider $\mathcal{O}\subseteq\{\Omega_2,\ldots,\Omega_n\}$ such that for each $\Omega_j\in\mathcal{O}$ there is a class V_i covered in Ω_j and in no other $\Omega_\ell\in\mathcal{O}$. Denote by $\mathcal{P}\subseteq\mathcal{O}$ the set of umbrellas in $\mathcal{O}$. We have*

$$|\Omega| \geq |\mathcal{U}|(n-|\mathcal{O}|) + \sum_{\Omega_j\in\mathcal{O}} |\Omega_j| - |\mathcal{U}|(|\mathcal{O}|-|\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1.$$

Proof. Let $W = \triangle_{U\in\mathcal{U}}U$. The number of edges in Ω is equal to $\sum_{j=1}^{n}\deg_\Omega(x_j)$. We bound $\deg_\Omega(x_j)$ by $|\mathcal{U}|$ for $j=1$ and if $\Omega_j\notin\mathcal{O}$ and by $|\Omega_j| - |\Omega_j\cap W|$ otherwise, see (iv) in Lemma 2. We obtain

$$|\Omega| \geq |\mathcal{U}|(n-|\mathcal{O}|) + \sum_{\Omega_j\in\mathcal{O}} \left(|\Omega_j| - |\Omega_j\cap W|\right).$$

We introduce a graph $G = (\mathcal{V},\mathcal{E})$ defined as follows. We use the terminology *nodes* and *links* for G in order to avoid confusion with the vertices and edges of Ω. The nodes in $\mathcal{V}$ are identified with the umbrellas in $\mathcal{U}$ and the Ω_j's in $\mathcal{O}$: $\mathcal{V} = \mathcal{U}\cup\mathcal{O}$. There is a link in $\mathcal{E}$ between two nodes if the corresponding octahedral systems have an edge in common. G is bipartite: indeed, two umbrellas in $\mathcal{U}$ are of the same colour V_{i_1} and, according to Proposition 2, they do not have an edge in common. According to Lemma 2, two Ω_j's do not have an edge in common either.

For Ω_j in $\mathcal{O}$, we have $|\Omega_j\cap W| = \sum_{U\in\mathcal{U}}|\Omega_j\cap U| = \deg_G(\Omega_j)$, note that here the degree is counted in G. The fact that the umbrellas in $\mathcal{U}$ are disjoint proves the first equality. The second inequality is deduced from the facts that Ω_j has at most one edge in common with each umbrella in $\mathcal{U}$, the one incident with x_j, and that Ω_j has no neighbours in $\mathcal{O}$. We obtain the following bound

$$\begin{aligned}
|\Omega| &\geq |\mathcal{U}|(n-|\mathcal{O}|) + \sum_{\Omega_j\in\mathcal{O}} \left(|\Omega_j| - \deg_G(\Omega_j)\right) \\
&= |\mathcal{U}|(n-|\mathcal{O}|) + \sum_{\Omega_j\in\mathcal{O}} |\Omega_j| - \deg_G(\mathcal{O}\setminus\mathcal{P}) - \deg_G(\mathcal{P}).
\end{aligned}$$

Again, for the equality, we use the fact G is bipartite. The number of links in $\mathcal{E}$ incident with a node in $\mathcal{O}\setminus\mathcal{P}$ is at most $|\mathcal{U}|$. Hence, $\deg_G(\mathcal{O}\setminus\mathcal{P}) \leq |\mathcal{U}|(|\mathcal{O}|-|\mathcal{P}|)$. It remains to bound $\deg_G(\mathcal{P})$. Note that if U is an umbrella in $\mathcal{P}$, it is the only umbrella of its colour in

$\mathcal{P}$, otherwise it would contradict the property of $\mathcal{O}$. We now prove that there are no cycles induced by $\mathcal{P} \cup \mathcal{U}$ in G.

Suppose there is such a cycle $\mathcal{C}$ and consider an umbrella U of $\mathcal{P}$ in this cycle. Denote its colour by V_i and its neigbours in $\mathcal{C}$ by L and R. As G is simple, L and R are distinct. L and R are both in $\mathcal{U}$, and hence are of colour V_{i_1} and do not have an edge in common. Therefore $U \cap L$ and $U \cap R$ do not have an edge in common either, which implies that the ith component of the transversals of L and R are distinct. Note that two umbrellas adjacent in $\mathcal{C}$, both of colour distinct from V_i, have necessarily transversals with the same ith component. Hence there must be another umbrella of colour V_i in the path in $\mathcal{C}$ between L and R not containing U. This is a contradiction since U is the only umbrella in $\mathcal{P}$ of colour V_i.

The number of links in $\mathcal{E}$ incident with $\mathcal{P}$ is then at most $|\mathcal{U}| + |\mathcal{P}| - 1$. This allows us to conclude. $\square$

Proof of Theorem 2. Let $\Omega \subseteq V_1 \times \cdots \times V_n$ be an octahedral system with $|V_1| = \cdots = |V_n| = n \geq 2$, and suppose that $k \geq 1$ classes $V_{i_1}, \ldots, V_{i_k}$, with $i_1 < \cdots < i_k$, are covered in Ω. The proof works by induction on k.

If $k = 1$, then Ω must contain at least n edges for one class to be covered.

Assume now that $k > 1$. If $|\mathcal{U}| \geq n - 1$, then, according to (iv) of Lemma 2, $|\Omega| = \sum_{j=1}^{n} \deg_\Omega(x_j) \geq n|\mathcal{U}| \geq k(n-2) + 2$ and we are done. Assume now that $|\mathcal{U}| \leq n - 2$. We consider a suitable decomposition $(\mathcal{U}, \Omega_2, \ldots, \Omega_n)$ of Ω and distinguish two cases.

Case 1: *One of the covered classes V_i, for $i \in \{i_2, \ldots, i_k\}$, is not covered in any Ω_j.* Let V_i be a covered class in Ω, while not being covered in any Ω_j. For each $j \in \{2, \ldots, n\}$, applying Lemma 3 on Ω_j gives a set $\mathcal{D}_j$ of umbrellas, all of colour distinct from V_i, such that $\Omega_j = \triangle_{U \in \mathcal{D}_j} U$. We obtain $\Omega = (\triangle_{U \in \mathcal{U}} U) \triangle (\triangle_{j=2}^{n} \triangle_{U \in \mathcal{D}_j} U)$, according to (ii) of Lemma 2. Thus, we can apply Proposition 3 which ensures that

$$|\Omega| \geq n^2 \geq k(n-2) + 2.$$

Case 2: *Each covered class V_i, for $i \in \{i_2, \ldots, i_k\}$, is covered in at least one of the Ω_j.* Choose a set $\mathcal{O} \subseteq \{\Omega_2, \ldots, \Omega_n\}$, minimal for inclusion, such that each covered class V_i, for $i \in \{i_2, \ldots, i_k\}$, is covered in at least one of the $\Omega_j \in \mathcal{O}$. Such a set $\mathcal{O}$ satisfies the statement of Proposition 4. Applying this proposition, we obtain

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} |\Omega_j| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1.$$

We now bound $\sum_{\Omega_j \in \mathcal{O}} |\Omega_j|$. By minimality of $\mathcal{O}$, there is at least one class covered in each $\Omega_j \in \mathcal{O}$. By induction, the cardinality of Ω_j is at least $k_j(n-2) + 2$, where $k_j \geq 1$ is the number of covered classes in Ω_j. We have $k_j < k$ according to (v) of Lemma 2. This lower bound is not good enough for the $\Omega_j \notin \mathcal{P}$ such that $k_j = 1$. We denote by $\mathcal{A}$ those Ω_j's. We explain now how to improve the lower bound for $\Omega_j \in \mathcal{A}$. Only one class is covered in Ω_j and $\Omega_j \notin \mathcal{P}$. According to Lemma 3, Ω_j can be written as a symmetric difference of distinct umbrellas of the same colour. According to Proposition 2, these umbrellas are pairwise disjoint and $|\Omega_j|$ is equal to n times the number of umbrellas in this decomposition. Since Ω_j is not an umbrella itself, otherwise Ω_j would have been in $\mathcal{P}$, there are at least two umbrellas in this decomposition. We obtain

$$\sum_{\Omega_j\in\mathcal{O}}|\Omega_j| \geq \left(\sum_{\Omega_j\in\mathcal{O}\setminus\mathcal{A}} k_j\right)(n-2)+2|\mathcal{O}\setminus\mathcal{A}|+2n|\mathcal{A}| = \left(\sum_{\Omega_j\in\mathcal{O}} k_j\right)(n-2)+2|\mathcal{O}|+n|\mathcal{A}|$$

We have thus

$$|\Omega| \geq |\mathcal{U}|(n-|\mathcal{O}|)+\left(\sum_{\Omega_j\in\mathcal{O}} k_j\right)(n-2)+2|\mathcal{O}|+n|\mathcal{A}|-|\mathcal{U}|(|\mathcal{O}|-|\mathcal{P}|)-|\mathcal{U}|-|\mathcal{P}|+1.$$

Finally, we have

$$2|\mathcal{O}|-|\mathcal{P}|-|\mathcal{A}| \leq \sum_{\Omega_j\in\mathcal{O}} k_j \tag{1}$$

$$k-1 \leq \sum_{\Omega_j\in\mathcal{O}} k_j \tag{2}$$

Equation (1) is obtained by distinguishing the Ω_j with $k_j=1$ from those with $k_j\geq 2$. Equation (2) results from the fact that each class $V_{i_2},\ldots,V_{i_k}$ is covered in at least one Ω_j in $\mathcal{O}$. Thus,

$$\begin{aligned}|\Omega| &\geq |\mathcal{U}|(n-|\mathcal{O}|)+\left(\sum_{\Omega_j\in\mathcal{O}} k_j\right)(n-2)+2|\mathcal{O}|+|\mathcal{U}||\mathcal{A}|-|\mathcal{U}|(|\mathcal{O}|-|\mathcal{P}|)-|\mathcal{U}|-|\mathcal{P}|+1\\ &\geq (k-1)(n-2)+2|\mathcal{O}|-|\mathcal{P}|+1+\left(\sum_{\Omega_j\in\mathcal{O}} k_j-k+|\mathcal{A}|+n-2|\mathcal{O}|+|\mathcal{P}|\right)|\mathcal{U}|\end{aligned}$$

where we only used the inequalities $n\geq n-2\geq|\mathcal{U}|$ and (2). According to (1), the expression $\left(\sum_{\Omega_j\in\mathcal{O}} k_j-k+|\mathcal{A}|+n-2|\mathcal{O}|+|\mathcal{P}|\right)$ is nonnegative. Moreover, we have already noted that $|\mathcal{U}|=\deg_\Omega(x_1)$, which is at least 1. Therefore,

$$|\Omega|\geq(k-1)(n-2)+2|\mathcal{O}|-|\mathcal{P}|+1+\sum_{\Omega_j\in\mathcal{O}} k_j-k+|\mathcal{A}|+n-2|\mathcal{O}|+|\mathcal{P}|.$$

Using (2) again, we obtain

$$|\Omega|\geq k(n-2)+2.$$

$\square$

Aknowlegement

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